

# SKIN EFFECT PROBLEM WITH THE DISPLACEMENT CURRENT IN MAXWELL PLASMA BY THE SOURCE METHOD

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We found the analytical solution to the problem of the skin effect for Maxwell plasma with the use of the kinetic equation, where the frequency of electron collisions is constant. We use the specular reflection of electrons from the surface as a boundary condition. The behavior of an impedance near to a plasma resonance is considered. We consider limiting cases of skin effect.

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**Introduction.** Skin effect is the plasma response to variable electromagnetic field, tangential to the surface [1, 2].

First, the analytical solution of the skin effect problem at any value of anomaly parameter was found in [3] and [4] for plasma in metal. For the gas plasma, the corresponding solution is considered in [5]. There has been substantial interest to this problem [6, 7, 8, 9, 10]. In [5], the behavior of plasma near the resonance is not considered. Also, research of impedance with displacement current near to resonance isn't carried out in the previous works. For example, the displacement current isn't taken into account in [2]. Research of surface impedance near to plasma resonance isn't carried out in the rest of works. We research the behaviour of impedance near to plasma resonance with the displacement current in this paper.

In this paper we continue the development of an analytical method of solving boundary problem for systems of equations for the electric field in half-space gas plasma. The basis of the method is the idea of the symmetrical continuation of the electric field to the conjugate half-space. We provide the analytical solution of the boundary problem of the skin effect theory for

electron plasma, that fills the half-space. We formulate analytical expressions for electric field, distribution function of electrons and impedance.

We assume that electromagnetic wave is incident normally to the interface of the plasma. In such configuration, the electric field of electromagnetic wave has only tangential component. We use the specular electron reflection from interface as boundary condition. The interface of ions on the conductivity of plasma is not considered.

### 1. Problem statement and basic equations.

Let Maxwell plasma fills the half-space  $x > 0$ , where  $x$  is the coordinate orthogonal to plasma boundary. Let the external electric field has only  $y$  component. Then the self-consistent electric field inside in plasma also has only  $y$  component  $E_y(x, t) = E(x)e^{-i\omega t}$ . Let us take the kinetic equation for distribution function of electrons:

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + eE(x)e^{-i\omega t} \frac{\partial f}{\partial p_y} = \nu(f_0 - f(t, x, \mathbf{v})). \quad (1)$$

In (1)  $\nu$  is the frequency of electron collisions with ions,  $e_0$  is the charge of electron,  $f_0(\mathbf{v})$  is the equilibrium Maxwell distribution function,

$$f_0(\mathbf{v}) = n \left( \frac{\beta}{\pi} \right)^{3/2} \exp(-\beta^2 \mathbf{v}^2), \quad \beta = \frac{m}{2k_B T}.$$

Here  $m$  is the mass of electron,  $k_B$  is the Boltzmann constant,  $T$  is the temperature of plasma,  $\mathbf{v}$  is the velocity of the electron,  $n$  is the concentration of electrons,  $c$  is the speed of light.

The electric field  $E(x)$  satisfies Poisson's equation

$$E''(x) + \frac{\omega^2}{c^2} E(x) = -\frac{4\pi i e^{i\omega t} \omega e}{c^2} \int v_y f(t, x, \mathbf{v}) d^3 v. \quad (2)$$

Assume that the intensity of an electric field is such that linear approximation is valid. Then distribution function can be represented in the form:

$$f = f_0 (1 + C_y \exp(-i\omega t) h(x, \mu)),$$

where  $\mathbf{C} = \sqrt{\beta} \mathbf{v}$  is the dimensionless velocity of electron,  $\mu = C_x$ . Let  $l = v_T \tau$  be the mean free path of electrons,  $v_T = 1/\sqrt{\beta}$ ,  $\tau = 1/\nu$ . We introduce the

dimensionless values:

$$t_1 = \nu t, \quad x_1 = \frac{x}{l}, \quad e(x_1) = \frac{\sqrt{2}e}{\nu\sqrt{mk_BT}}E(x_1).$$

Below, instead of  $x_1$  we shall write again  $x$ . In new variables, the kinetic equation (1) and the equation on a field with the displacement current (2) become

$$\mu \frac{\partial h}{\partial x} + z_0 h(x, \mu) = e(x), \quad z_0 = 1 - i\omega\tau, \quad (3)$$

$$e''(x) + Q^2 e(x) = -i \frac{\alpha}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\mu'^2) h(x, \mu') d\mu', \quad Q = \frac{\omega l}{c}, \quad (4)$$

where  $\delta = \frac{c^2}{2\pi\omega\sigma_0}$  is the classical depth of the skin layer,  $\sigma_0 = \frac{e^2 n}{m\nu}$ ,  $\alpha = \frac{2l^2}{\delta^2}$ , what  $\alpha$  as the anomaly parameter.

We formulate the boundary conditions for the distribution function of the electron in case of the specular electron reflection from the surface:

$$h(0, \mu) = h(0, -\mu), \quad 0 < \mu < +\infty. \quad (5)$$

We use the condition that function  $h(x, \mu)$  vanishes far from the surface:

$$h(+\infty, \mu) = 0, \quad -\infty < \mu < +\infty, \quad (6)$$

and conditions for electric field on the interface and far from it:

$$e'(0) = e_s', \quad e(+\infty) = 0, \quad (7)$$

where  $e_s'$  is the given value of gradient of electric field on the plasma interface.

So, the skin effect problem is formulated completely. We seek solution of system of the equations (3) and (4) in this problem that satisfy boundary conditions (5)–(7).

**2. The analytical solution of the problem.** As a first step in the source method, we extend the electric field and distribution function to the "negative" half-space  $x < 0$ :

$$e(x) = e(-x), \quad h(x, \mu) = h(-x, -\mu). \quad (8)$$

After we substitute  $x = 0$  to equation (8), we obtain that electric field and distribution function of electrons are continuous and the derivative of

an electric field has discontinuity:  $e'(+0) - e'(-0) = 2e'_s$ . In account of this circumstance, we introduce term with Dirac delta function to the field equation [10]:

$$e''(x) + Q^2 e(x) - 2e'_s \delta(x) = -i \frac{\alpha}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\mu'^2) h(x, \mu') d\mu', \quad (9)$$

where  $\delta(x)$  is the Dirac delta function.

The third term in the left hand side of the equation (9) corresponds to discontinuity of a derivative of the electric field for  $x = 0$ .

The solution of the (3), (9), (5)–(7) can be sought as Fourier integrals (by variable  $x$ ):

$$e(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} E(k) dk, \quad (10)$$

$$h(x, \mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \Phi(k, \mu) dk, \quad (11)$$

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk. \quad (12)$$

We substitute (10)–(12) into (3) and (9). We get the following system of the characteristic equations:

$$(Q^2 - k^2) E(k) - 2e'(0) = -i \frac{\alpha}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\mu^2) \Phi(k, \mu) d\mu,$$

$$\Phi(k, \mu)(z_0 + ik\mu) = E(k).$$

From these equations, we get spectral densities of the distribution function and electric field, respectively:

$$\Phi(k, \mu) = \frac{E(k)}{ik\mu + z_0}, \quad (13)$$

$$E(k) = -\frac{2e'_s}{k^2 \lambda(k)}, \quad (14)$$

where

$$\lambda(k) = 1 - \frac{Q^2}{k^2} - i \frac{\alpha}{k^2 \sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\exp(-\mu^2) d\mu}{ik\mu + z_0}.$$

To find the profile of the electric field in the half-space, we substitute (14) into (10):

$$e(x) = -\frac{e'_s}{\pi} \int_{-\infty}^{\infty} \frac{e^{ikx} dk}{k^2 \lambda(k)}. \quad (15)$$

To get the function of the electron distribution in the half-space we substitute (14) into (13). It is obvious that this spectral density is:

$$\Phi(k, \mu) = -\frac{2e'_s}{(z_0 + ik\mu)k^2 \lambda(k)}. \quad (16)$$

Now we substitute (16) into (11). We get

$$h(x, \mu) = -\frac{e'_s}{\pi} \int_{-\infty}^{\infty} \frac{e^{ikx} dk}{(z_0 + ik\mu)k^2 \lambda(k)}.$$

**3. The impedance evaluation.** We introduce the dimensionless decrease of the electric field into the depth of plasma:

$$\Lambda(\alpha, \Omega) = -\frac{e(0)}{e'_s}, \quad \Omega = \omega\tau. \quad (17)$$

From (15) this dimensionless decrement is:

$$\Lambda(\alpha, \Omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dk}{k^2 \lambda(k)} = \frac{2}{\pi} \int_0^{\infty} \frac{dk}{k^2 \lambda(k)}.$$

The value of the impedance can be calculated with the help of the formula (see [1]):

$$Z = \frac{4\pi i \omega}{c^2} \frac{e(x)}{\left. \frac{de(x)}{dx} \right|_{x=0}}.$$

It should be noted that

$$\frac{de(x')}{dx'} = \frac{de(x)}{dx} \cdot \frac{dx}{dx'} = l \frac{de(x)}{dx}.$$

Thus, according to the previous formula the impedance is

$$Z = \frac{4\pi i \omega l}{c^2} \cdot \frac{e(0)}{e'_s},$$

or, taking into consideration the equality (17),

$$Z = -i \frac{4\pi \omega l}{c^2} \Lambda(\alpha, \Omega). \quad (18)$$

Let us introduce parameter  $R$ , which is equal to the modulus of impedance in normal skin effect (when  $\alpha \ll 1$ ,  $\Omega \ll 1$ )

$$R = \sqrt{\frac{4\pi\omega}{c^2\sigma_0}},$$

where  $\sigma_0$  is the static electrical conductivity of plasma (for  $\omega = 0$ ). Now formula (18) for the impedance can be written in the form  $Z = RZ_0$ , where  $Z_0$  is the dimensionless part of impedance,

$$Z_0 = -i\sqrt{\alpha}\Lambda(\alpha, \Omega). \quad (19)$$

**4. The analysis of the solution.** We substitute the variable  $k = 1/t$  to the integral (19). In this case the dimensionless decrement is:

$$\Lambda(\alpha, \Omega) = \frac{2}{\pi} \int_0^\infty \frac{dt}{\lambda(1/t)}.$$

Here

$$\lambda(1/t) = 1 - Q^2 t^2 - \frac{\alpha t^3}{\sqrt{\pi}} \int_{-\infty}^\infty \frac{\exp(-\mu^2) d\mu}{\mu - iz_0 t}.$$

We will present special partial cases of formula (19).

We will start from the case of the normal skin effect. In this case

$$\alpha \ll 1, \quad \Omega \ll 1, \quad z_0 = 1 - i\omega \approx 1.$$

We suppose that there is no current of the bias. In this case the dimensionless impedance depends on  $\alpha$  is

$$Z_0 = -i\sqrt{\alpha} \frac{2}{\pi} \int_0^\infty \frac{dt}{1 - \alpha t^3 \frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty \frac{\exp(-u^2) du}{u - it}},$$

or, if we introduce

$$t_0(iz) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty \frac{\exp(-u^2) du}{u - iz},$$

the expression for the dimensionless impedance will be follows:

$$Z_0 = -i\sqrt{\alpha} \frac{2}{\pi} \int_0^\infty \frac{dt}{1 - \alpha t^3 t_0(it)}.$$

It should be noted that for the large  $t$ :  $t_0(it) \approx i/t$ . Therefore

$$Z_0 = -i\sqrt{\alpha} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dt}{1 - i\alpha t^2} = 2\sqrt{\alpha} \cdot \operatorname{Res}_{z=-\frac{1-i}{\sqrt{2\alpha}}} \frac{1}{1 - i\alpha z^2} = \frac{1-i}{\sqrt{2}}.$$

The expression  $Z_0 = \frac{1-i}{\sqrt{2}}$  is a well-known classical result [5].

We will consider the anomalous skin effect in the low-frequency limit, that is, when  $\alpha \gg 1$ ,  $\Omega \ll 1$ ,  $z_0 \approx 1$ . In this case for small  $t$  we have:

$$t_0(it) \approx \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\exp(-u^2) du}{u - it} \approx i\sqrt{\pi} \exp(-(it)^2) \approx i\sqrt{\pi}.$$

Thus the non dimensional impedance is

$$Z_0 = -i\sqrt{\alpha} \frac{2}{\pi} \int_0^{\infty} \frac{dt}{1 - i\alpha \sqrt{\pi} t^3}.$$

We substitute the variable in this integral  $t = 1/k$  and obtain

$$Z_0 = -i\sqrt{\alpha} \frac{2}{\pi} \int_0^{\infty} \frac{k dk}{k^3 - i\alpha \sqrt{\pi}}.$$

We substitute one more variable  $k = \sqrt[3]{\alpha \sqrt{\pi}} x$ , so

$$Z_0 = -i\sqrt{\alpha} \frac{2\sqrt[3]{\alpha \sqrt{\pi}}}{\pi} \int_0^{\infty} \frac{x dx}{x^3 - i}.$$

We consider the integral

$$J = \int_0^{\infty} \frac{x dx}{x^3 - i} = J_1 + iJ_2, \quad J_1 = \int_0^{\infty} \frac{x^4 dx}{x^6 + 1}, \quad J_2 = \int_0^{\infty} \frac{x dx}{x^6 + 1}.$$

We will calculate the integral  $J_1$  with the use of residues. The function under integral has simple poles in the points  $z_k = \exp\left(\frac{i\pi}{6}(1 + 2k)\right)$ ,  $k = 0, 1, 2$ . So

$$J_1 = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^4 dx}{x^6 + 1} = \pi i \sum_{k=0}^1 \operatorname{Res}_{z=z_k} \frac{z^4}{z^6 + 1} = \frac{\pi i}{6} \left( \frac{1}{z_0} + \frac{1}{z_1} + \frac{1}{z_2} \right) = \frac{\pi}{3}.$$

The second integral is calculated by the decomposition of the function under integral into elementary fractions. As a result we have:  $J_2 = \frac{\pi}{3\sqrt{3}}$ . Thus,  $J = \frac{\pi(\sqrt{3} + i)}{3\sqrt{3}}$ , and the expression for the non dimensional impedance is

$$Z_0 = \frac{2\sqrt[6]{\alpha}}{3\sqrt{3}}(1 - i\sqrt{3}),$$

which also coincides with the classical result [5].

Further, to study the impedance near to a plasma resonance, it is convenient to use the following dimensionless parameters:

$$\gamma = \frac{\omega}{\omega_p}, \quad \varepsilon = \frac{\nu}{\omega_p}, \quad \text{where} \quad \omega_p = \frac{4\pi e_0^2 n}{m}.$$

Here  $\omega_p$  is the plasma frequency.

We express the parameters of the problem  $\alpha, \Omega, Q$  through  $\gamma, \varepsilon$  and  $v_c = v_T/c$ , where  $v_T = 1/\sqrt{\beta}$  is the heat velocity of the electrons. We obtain that

$$\alpha = \frac{\gamma v_c^2}{\varepsilon^3}, \quad Q = \frac{\gamma v_c}{\varepsilon}, \quad \Omega = \frac{\gamma}{\varepsilon}.$$

We analyze numerically the growth of value of the modulus of impedance, the real, imaginary parts of impedance and argument of impedance depending on change of value  $\gamma$  from 0.5 to 1.2 with various values of other parameters. If we change of the anomaly parameter  $\alpha$  in the specified limits, the value  $\gamma$  becomes  $\gamma = 1$ , that is  $\omega = \omega_p$  i.e. the oscillation frequency of external field is the value of plasma frequency. This is plasma resonance. It would be interesting to consider parameters of self-concordant field near plasma resonance. In conclusion we show the results of numerical analysis.

**Conclusion.** The analysis of plots in figure 1a shows that at the same temperature of plasma, the maximum of the modulus of impedance is reached at  $\gamma = 1$ , i.e. for  $\omega = \omega_p$ . Thus, the less is the effective frequency of electrons collisions with particles of plasma, the greater is the modulus of impedance.

From figure 1b we see that at the same frequency of collisions of electrons the maximum of the modulus of impedance is reached at  $\omega = \omega_p$ , independent of the temperature. Actual curves in this figure are computed for various values of parameter  $v_c = v_T/c$ , which depends on temperature (it is proportional



to root square of temperature):  $v_c = \sqrt{2k_B T / c^2 m}$ . Thus the size of the modulus of impedance decreases quickly with growth of temperature.

It is interesting to note that for decrease reduction of value  $\varepsilon$  from  $10^{-2}$  to  $10^{-4}$  (by two orders), the modulus of impedance also increases by two orders, more precisely 95 times. The temperature of the plasma here is  $3000K$ . If temperature of the plasma is  $5000K$ , the modulus of impedance increases by 97 times for the same reduction in the value of  $\varepsilon$ .

For an increase in the temperature of the plasma from  $3000K$  ( $v_c = 10^{-3}$ ) to  $5000K$  ( $v_c = 13 \cdot 10^{-3}$ ) the value of the modulus of impedance increases by 170 times, and for a change in the temperature of the plasma from  $1000K$  to  $3000K$ , the value of the modulus of impedance changes only by a factor of 2.8. Thus, the growth of the modulus of impedance depends on the nonlinear change in temperature.

In figures 2a and 2b we show the plots of the real part of impedance (more precisely, the plots of values  $\text{Re}(-Z_0)$ ). The analysis of plots in figures 2 shows that as the frequency of collisions of electrons increases, the value  $\text{Re}(-Z_0)$  grows at a constant temperature. If the frequency of collisions of electrons is constant, then this value grows as the temperature increase.

Let us note, that in figures 1 and 2 we used the logarithmic scale for vertical axes.

The analysis of dependence of argument of impedance on parameter  $\gamma$  shows that near to plasma resonance, the argument of impedance has step irrespective of the frequency of electron collisions (fig. 3) and from temperature.

So, the numerical analysis of plots (fig. 1, 2) shows that near to plasma resonance, the modulus and imaginary part of impedance have the sharp maximum, which is absent in low-frequency limit, or in the normal skin effect theory, where the argument near to resonance has step, and the real part of impedance has the sharp maximum.

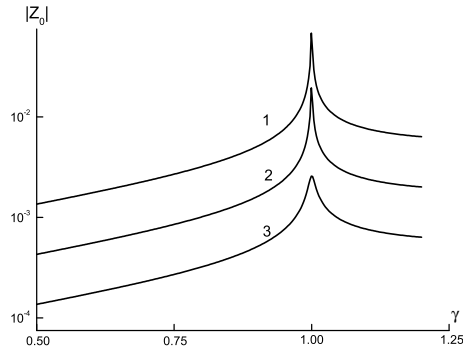


Fig. 1a. The modulus of impedance. For the curves 1, 2, 3,  $\varepsilon = 10^{-4}, 10^{-3}, 10^{-2}$  respectively; and  $v_c = 10^{-3}$ .

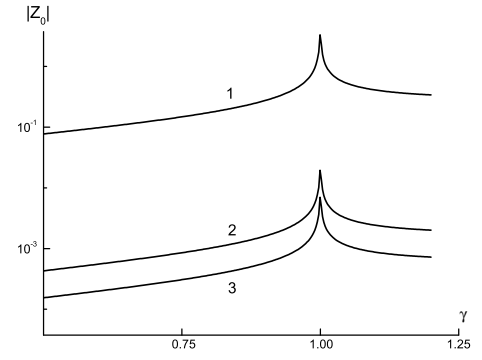


Fig. 1b. The modulus of impedance. For the curves 1, 2, 3,  $v_c = 6 \cdot 10^{-4}, 10^{-3}, 13 \cdot 10^{-3}$  respectively; and  $\varepsilon = 10^{-3}$ .

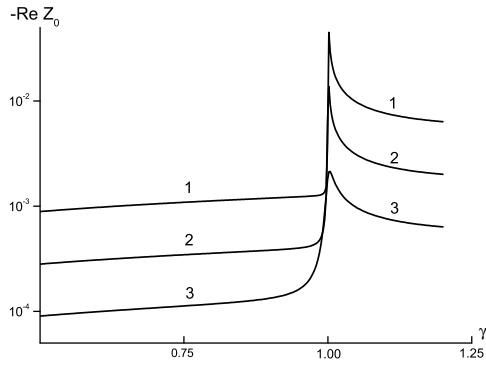


Fig. 2a. The real part of impedance. For the curves 1, 2, 3,  $\varepsilon = 10^{-4}, 10^{-3}, 10^{-2}$  respectively; and  $v_c = 10^{-3}$ .

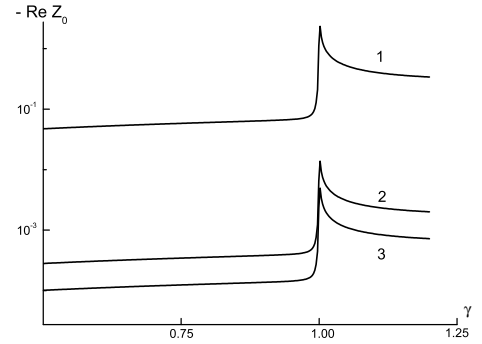


Fig. 2b. The real part of impedance. For the curves 1, 2, 3,  $v_c = 13 \cdot 10^{-3}, 10^{-3}, 6 \cdot 10^{-4}$  respectively; and  $\varepsilon = 10^{-3}$ .

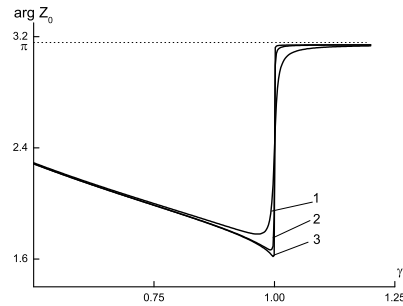


Fig. 3. The argument of impedance. For the curves 1, 2, 3,  $\varepsilon = 10^{-2}, 10^{-3}, 10^{-4}$  respectively; and  $v_c = 10^{-3}$ .

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